

QUANDLE AND HYPERBOLIC VOLUME

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ABSTRACT. We show that the hyperbolic volume of a hyperbolic knot is a quandle cocycle invariant. Further we show that it completely determines invertibility and positive/negative amphicheirality of hyperbolic knots.

1. INTRODUCTION

A quandle introduced by D. Joyce [9] and S. V. Matveev [11] independently, is an algebraic system having a self-distributive binary operation whose definition is motivated by knot theory. They defined the knot quandle, and showed that it completely classifies knots. J. S. Carter et al. have developed a theory of quandle cocycle invariants in [2]. Several useful applications of quandle homology/cohomology theory have been established; distinguishing the unknot [3], determining non-invertibility of classical/surface knots [2, 13, 14], and estimating the minimal triple point number of a surface knot [15], for examples. However, there seems to be no conceptual understanding of quandle cocycle invariants so far.

In this paper, we would like to present such one by showing that there is a quandle cocycle invariant whose each element is 1, -1 or 0 times volume for the hyperbolic knots (Theorem 3.3). Further we show that it completely determines invertibility and positive/negative amphicheirality of hyperbolic knots (Theorem 4.1, 4.3, and 4.4).

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2. PRELIMINARIES

2.1. Knot quandle. In this subsection, we briefly recall the definition of a quandle and the knot quandle. See [4, 9, 11] for examples for more details.

A *quandle* is defined to be a set Q with a binary operation $*$ on Q satisfying the following properties:

- (Q1) For each $x \in Q$, $x * x = x$.
- (Q2) For each $y \in Q$, the map $*y : Q \rightarrow Q$ ($x \mapsto x * y$) is bijective.
- (Q3) For each triple $x, y, z \in Q$, $(x * y) * z = (x * z) * (y * z)$.

For example, if we define a binary operation $*$ on a subset X of a group G closed under conjugations by

$$a * b = b^{-1}ab \quad (\forall a, b \in X)$$

then X together with $*$ becomes a quandle. We call it the *conjugation quandle*.

Suppose that K is an oriented prime knot in S^3 . It is easy to see that the set $\mathcal{Q}(K)$ of positive meridians of $\pi_1(S^3 \setminus K)$, which are oriented meridians compatible with the orientation of the knot, is closed under conjugations. The *knot quandle* of K is defined to be its conjugation quandle.

2.2. Quandle cocycle invariant. In this subsection, we briefly recall the definition of a quandle cocycle invariant. See [1, 2, 5, 6, 10] for examples for more details.

Let $\mathcal{F}(Q)$ be the free group on Q and $\mathcal{N}(Q)$ the subgroup of $\mathcal{F}(Q)$ normally generated by

$$y^{-1}xy(x*y)^{-1} \quad (\forall x, y \in Q).$$

We denote the quotient group $\mathcal{F}(Q)/\mathcal{N}(Q)$ by $\mathcal{G}(Q)$.

Suppose that D is a diagram of an oriented knot K . An *arc coloring* of D is defined to be a map

$$\mathcal{A} : \{\text{arcs of } D\} \longrightarrow Q$$

satisfying the condition illustrated in the left-hand side of Figure 1 at each crossing point. Further a *region coloring* of D is defined to be a map

$$\mathcal{R} : \{\text{regions of } D\} \longrightarrow Y,$$

where Y is a set equipped with a right action of $\mathcal{G}(Q)$, satisfying the condition depicted in the right-hand side of Figure 1 around each arc. We call a pair $(\mathcal{A}, \mathcal{R})$ a *shadow coloring* of D , and denote by \mathcal{S} .

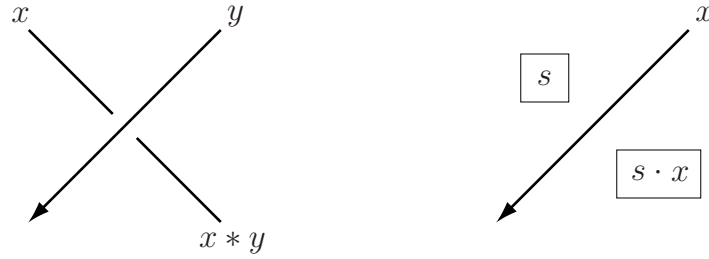


FIGURE 1. Rules for colorings

Choose an abelian group A . An A -valued *quandle 2-cocycle* with respect to Q and Y is defined to be a map

$$\theta : Y \times Q \times Q \longrightarrow A$$

satisfying the following conditions:

- (i) For each $r \in Y$ and $x \in Q$, $\theta(r, x, x) = 0$.
- (ii) For each $r \in Y$ and $x, y, z \in Q$,

$$\begin{aligned} & \theta(r, x, y) + \theta(r \cdot y, x * y, z) + \theta(r, y, z) \\ &= \theta(r \cdot x, y, z) + \theta(r, x, z) + \theta(r \cdot z, x * z, y * z). \end{aligned}$$

For each crossing point c of D , a Boltzmann weight of c is defined as

$$B(\mathcal{S}, \theta, c) = \varepsilon(c) \theta(r, x, y),$$

where $\varepsilon(c)$ is 1 or -1 depending on whether c is positive or negative respectively, and $r \in Y$ and $x, y \in Q$ denote colors around c as depicted in Figure 2. Further we let

$$\Phi(\mathcal{S}, \theta) = \sum_{c \in \mathcal{C}} B(\mathcal{S}, \theta, c),$$

where \mathcal{C} denotes the set of crossing points of D .

Theorem 2.1 ([2, 5, 10]). *The multi-set*

$$\{\Phi(\mathcal{S}, \theta) \in A \mid \mathcal{S} \text{ is a shadow coloring of } D \text{ with respect to } Q \text{ and } Y\}$$

does not depend on the choice of a diagram D of K .

We call this multi-set a *quandle cocycle invariant* of K .

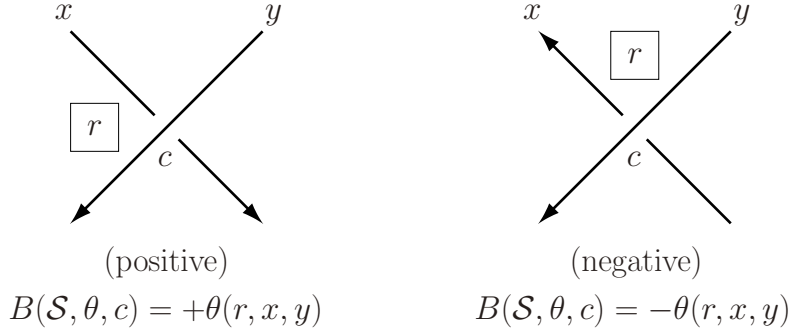


FIGURE 2. Boltzmann weight of a positive/negative crossing

3. HYPERBOLIC VOLUME IS QUANDLE COCYCLE INVARIANT

Let K be an oriented hyperbolic knot in S^3 ,

$$\Psi : \mathbb{H}^3 \longrightarrow S^3 \setminus K$$

the universal covering, $p \in S^3 \setminus K$ a base point of $\pi_1(S^3 \setminus K)$, and $\tilde{p} \in \Psi^{-1}(p)$. Then we have a holonomy representation

$$\rho : \pi_1(S^3 \setminus K) \longrightarrow \text{Isom}_+ \mathbb{H}^3.$$

For each positive meridian $x \in \mathcal{Q}(K)$, remarking that $\rho(x)$ is parabolic, we denote the fixed point of $\rho(x)$ on $\partial \overline{\mathbb{H}^3} = S_\infty^2$ by x_∞ .

Let $q \in S^3 \setminus K$ be a point other than p , and Z the set of homotopy classes of paths from p to q . Then Z admits the right action of $\mathcal{G}(\mathcal{Q}(K))$ by composing the inverse of a closed loop representing an element of $\mathcal{G}(\mathcal{Q}(K))$ by the left. For each $r \in Z$, \tilde{r} denotes a lift of a representative path of r satisfying $\tilde{r}(0) = \tilde{p}$.

For each $r \in Z$ and $x, y \in \mathcal{Q}(K)$, we define a 3-dimensional singular chain $C_{r,x,y}$ of $S^3 \setminus K$ as

$$\begin{aligned} C_{r,x,y} = & \{\tilde{p}, \tilde{r}(1), x_\infty, y_\infty\} + \{\tilde{p}, \widetilde{r \cdot x}(1), y_\infty, x_\infty\} \\ & + \{\tilde{p}, \widetilde{r \cdot xy}(1), (x * y)_\infty, y_\infty\} + \{\tilde{p}, \widetilde{r \cdot y}(1), y_\infty, (x * y)_\infty\}, \end{aligned}$$

where $\{v_0, v_1, v_2, v_3\}$ ($v_0, v_1, v_2, v_3 \in \overline{\mathbb{H}^3}$) denotes a singular simplex of $S^3 \setminus K$ defined as a map from the tetrahedron Δ^3 possibly with ideal vertices to the image of a geodesic tetrahedron spun by v_0, v_1, v_2 , and v_3 by Ψ , under the assumption that ideal vertices are properly understood. Further we define a map

$$\text{vol} : Z \times \mathcal{Q}(K) \times \mathcal{Q}(K) \longrightarrow \mathbb{R}$$

by

$$\begin{aligned} \text{vol}(r, x, y) = & \text{algvol}(\{\tilde{p}, \tilde{r}(1), x_\infty, y_\infty\}) \\ & + \text{algvol}(\{\tilde{p}, \widetilde{r \cdot x}(1), y_\infty, x_\infty\}) \\ & + \text{algvol}(\{\tilde{p}, \widetilde{r \cdot xy}(1), (x * y)_\infty, y_\infty\}) \\ & + \text{algvol}(\{\tilde{p}, \widetilde{r \cdot y}(1), y_\infty, (x * y)_\infty\}), \end{aligned}$$

where $\text{algvol}(\{v_0, v_1, v_2, v_3\})$ denotes the algebraic volume of a singular simplex $\{v_0, v_1, v_2, v_3\}$. It is convenient that we extend the domain of $\text{algvol}(\cdot)$ to singular chains linearly. Then $\text{vol}(r, x, y) = \text{algvol}(C_{r,x,y})$.

Proposition 3.1. *vol is an \mathbb{R} -valued quandle 2-cocycle with respect to $\mathcal{Q}(K)$ and Z .*

Proof. For each $r \in Z$ and $x \in \mathcal{Q}(K)$, it is obvious that each simplex constructing a singular chain $C_{r,x,x}$ degenerates. Thus

$$\text{vol}(r, x, x) = \text{algvol}(C_{r,x,x}) = 0.$$

For each $r \in Z$ and $x, y, z \in \mathcal{Q}(K)$, it is routine to check that several Pachner moves transform a singular chain $(C_{r,x,y} + C_{r \cdot y, x * y, z} + C_{r,y,z})$

into another singular chain $(C_{r \cdot x, y, z} + C_{r, x, z} + C_{r \cdot z, x * z, y * z})$. Thus

$$\begin{aligned}
& \text{vol}(r, x, y) + \text{vol}(r \cdot y, x * y, z) + \text{vol}(r, y, z) \\
& \quad - \text{vol}(r \cdot x, y, z) - \text{vol}(r, x, z) - \text{vol}(r \cdot z, x * z, y * z) \\
= & \text{algvol}(C_{r, x, y} + C_{r \cdot y, x * y, z} + C_{r, y, z}) \\
& \quad - \text{algvol}(C_{r \cdot x, y, z} + C_{r, x, z} + C_{r \cdot z, x * z, y * z}) \\
= & 0.
\end{aligned}$$

□

Proposition 3.2. *For each shadow coloring \mathcal{S} of a diagram D of K with respect to $\mathcal{Q}(K)$ and Z , there exists $k \in \mathbb{Z}$ such that*

$$\Phi(\mathcal{S}, \text{vol}) = k \cdot \text{vol}(S^3 \setminus K).$$

Here $\text{vol}(S^3 \setminus K)$ denotes the hyperbolic volume of $S^3 \setminus K$.

Proof. Since

$$\Phi(\mathcal{S}, \text{vol}) = \text{algvol}\left(\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}\right),$$

where $r \in Z$ and $x, y \in \mathcal{Q}(K)$ denote colors around a crossing point c , it is sufficient to prove that

$$\partial\left(\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}\right) = 0,$$

that is,

$$\left(\overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}}, \partial \overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}}\right) \in \mathbb{Z}_3(\overline{S^3 \setminus K}, \partial \overline{S^3 \setminus K}).$$

Here $\partial(\cdot)$ denotes the boundary operator, $\overline{S^3 \setminus K}$ a compactification of $S^3 \setminus K$ with a torus boundary, and $\left(\overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}}, \partial \overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}}\right)$ a relative singular chain with respect to a singular chain $\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}$ which is naturally defined by the compactification. At this time,

$$\left[\overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}}, \partial \overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}}\right] \in \mathbb{H}_3(\overline{S^3 \setminus K}, \partial \overline{S^3 \setminus K})$$

must be an integral multiple of the fundamental class, and thus

$$\text{algvol}\left(\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r,x,y}\right) = k \cdot \text{vol}(S^3 \setminus K).$$

By a straightforward calculation,

$$\begin{aligned} \partial C_{r,x,y} = & \{\widetilde{p}, \widetilde{r}(1), y_\infty\} - \{\widetilde{p}, \widetilde{r \cdot y}(1), y_\infty\} \\ & + \{\widetilde{p}, \widetilde{r \cdot x}(1), x_\infty\} - \{\widetilde{p}, \widetilde{r}(1), x_\infty\} \\ & + \{\widetilde{p}, \widetilde{r \cdot xy}(1), y_\infty\} - \{\widetilde{p}, \widetilde{r \cdot x}(1), y_\infty\} \\ & + \{\widetilde{p}, \widetilde{r \cdot y}(1), (x * y)_\infty\} - \{\widetilde{p}, \widetilde{r \cdot xy}(1), (x * y)_\infty\}, \end{aligned}$$

where $\{v_0, v_1, v_2\}$ ($v_0, v_1, v_2 \in \overline{\mathbb{H}^3}$) denotes a 2-dimensional singular simplex of $S^3 \setminus K$ defined as a map from the triangle Δ^2 possibly with ideal vertices to the image of a geodesic triangle spun by v_0, v_1 , and v_2 by Ψ . Further corresponding to each adjacent crossing points c and c' , singular chains $\varepsilon(c) \partial C_{r,x,y}$ and $\varepsilon(c') \partial C_{r',x',y'}$, where $r' \in Z$ and $x', y' \in \mathcal{Q}(K)$ denote colors around c' , have canceling terms as depicted in Figure 3. Thus

$$\partial \left(\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r,x,y} \right) = \sum_{c \in \mathcal{C}} \varepsilon(c) \partial C_{r,x,y} = 0.$$

□

$$\begin{aligned} & \varepsilon(c)(\{\widetilde{p}, \widetilde{r \cdot x}(1), x_\infty\} - \{\widetilde{p}, \widetilde{r}(1), x_\infty\}) \\ & + \varepsilon(c')(\{\widetilde{p}, \widetilde{r' \cdot x' y'}(1), y'_\infty\} - \{\widetilde{p}, \widetilde{r' \cdot x'}(1), y'_\infty\}) = 0 \end{aligned}$$

FIGURE 3. Adjacent crossing points cancel a pair of terms

Furthermore, we refine Proposition 3.2 as follows.

Theorem 3.3. *For each shadow coloring \mathcal{S} of a diagram D of K with respect to $\mathcal{Q}(K)$ and Z , there exists $k \in \{-1, 0, 1\}$ such that*

$$\Phi(\mathcal{S}, \text{vol}) = k \cdot \text{vol}(S^3 \setminus K).$$

To prove the theorem, we consider the following decomposition of $S^3 \setminus K$ introduced in [8]. Put a diagram D of K on S^2 which divides S^3 into two connected components containing p or q respectively. Take a dual graph of D on S^2 , and consider its suspension with respect to p and q . Then we have a decomposition of $S^3 \setminus K$ into thin regions like bananas illustrated in Figure 4. Further we cut each banana into four pieces as depicted in Figure 5.

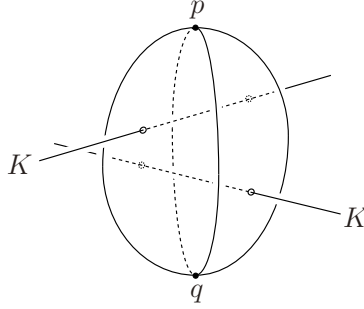


FIGURE 4. A banana

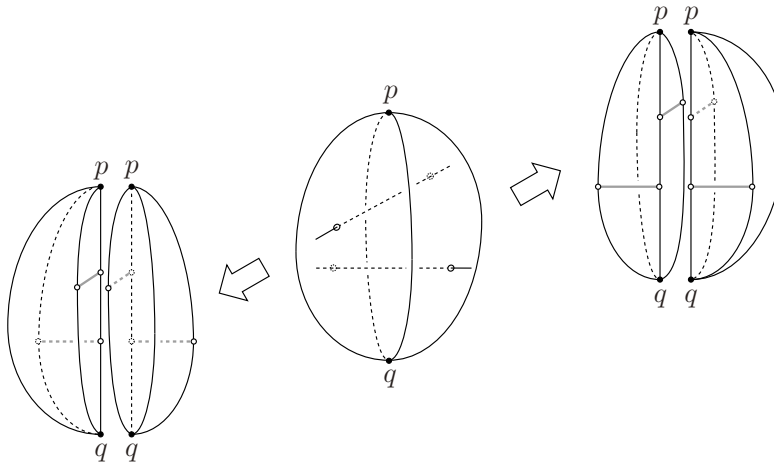


FIGURE 5. Cutting a banana into four pieces

Proof of Theorem 3.3. For each piece P of bananas, define a surjective continuous map τ from P to the tetrahedron Δ^3 with two ideal vertices as depicted in Figure 6 or 7 depending on the shape of P . Let x or y in $\mathcal{Q}(K)$ be the color of the arc a or b respectively, and $r \in Z$ the color of the region with which the edge e intersects. Then the composition $\{\tilde{p}, \tilde{r}(1), x_\infty, y_\infty\} \circ \tau$ is a continuous map from P to $S^3 \setminus K$. By the construction, modifying each τ step-by-step if necessary, we can assume that

$$\{\tilde{p}, \tilde{r}(1), x_\infty, y_\infty\} \circ \tau|_{\partial P \cap \partial P'} = \{\tilde{p}', \tilde{r}'(1), x'_\infty, y'_\infty\} \circ \tau'|_{\partial P \cap \partial P'}$$

for each pair of pieces P and P' of bananas, where τ' denotes a surjective continuous map from P' to Δ^3 , and $x', y' \in \mathcal{Q}(K)$ and $r' \in Z$ colors with respect to P' . Thus we have a continuous map

$$f_S : S^3 \setminus K \longrightarrow S^3 \setminus K$$

satisfying $f_S|_P = \{\tilde{p}, \tilde{r}(1), x_\infty, y_\infty\} \circ \tau$. By the assumption that K is hyperbolic, the degree of f_S must be 1, -1 or 0, or else the simplicial volume of $S^3 \setminus K$ must be 0 being untrue to the assumption. Thus

$$\left[\overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r,x,y}}, \partial \overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r,x,y}} \right] \in H_3(\overline{S^3 \setminus K}, \partial \overline{S^3 \setminus K})$$

must be 1, -1 or 0 times the fundamental class. \square

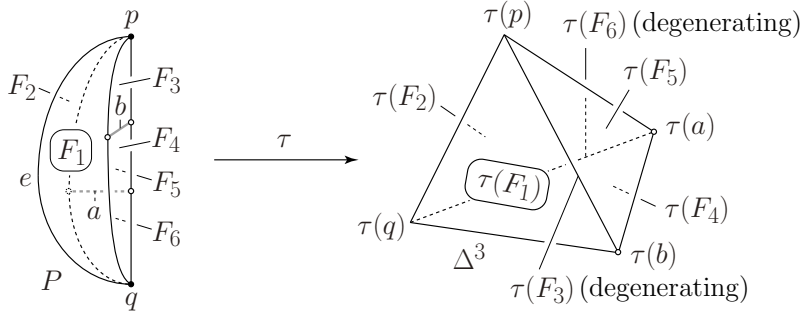
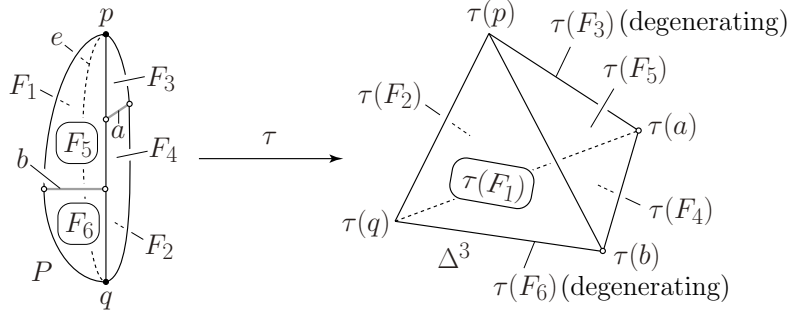


FIGURE 6. A surjective continuous map τ

Let \mathcal{A}_* be an arc coloring of a diagram D of K with respect to $\mathcal{Q}(K)$ mapping each arc of D to the Wirtinger generator (cf. Section 3.D. of [12]) with respect to the arc, \mathcal{R}_* a region coloring of D with respect to

FIGURE 7. Another surjective continuous map τ

Z mapping each region to the homotopy class of the edge of bananas which intersects with the region, and $\mathcal{S}_* = (\mathcal{A}_*, \mathcal{R}_*)$.

Theorem 3.4.

$$\Phi(\mathcal{S}_*, \text{vol}) = \text{vol}(S^3 \setminus K).$$

Proof. For each piece P of bananas, $\{\tilde{p}, \tilde{r}(1), x_\infty, y_\infty\} \circ \tau$ is homotopic to the identity map of P , because $\{\tilde{p}, \tilde{r}(1), x_\infty, y_\infty\} \circ \tau$ corresponds to the straightening of the identity map. Thus $f_{\mathcal{S}_*}$ is homotopic to the identity map of $S^3 \setminus K$. \square

4. DETERMINING INVERTIBILITY AND AMPHICHEIRALITY

Let K be an oriented hyperbolic knot in S^3 . We denote K with reversed orientation by $-K$, and a mirror image of K by K^* . $-K^*$ is thus a mirror image of K with reversed orientation.

Theorem 4.1. *K is equivalent to $-K^*$ if and only if there exists a shadow coloring \mathcal{S} of a diagram D of K with respect to $\mathcal{Q}(K)$ and Z satisfying*

$$\Phi(\mathcal{S}, \text{vol}) = -\text{vol}(S^3 \setminus K).$$

Proof. First, we show “if” part. By Thurston’s rigidity theorem (cf. [7] for example), there is an orientation reversing homeomorphism f of $S^3 \setminus K$ being homotopic to $f_{\mathcal{S}}$. Since $f_{\mathcal{S}}$ maps each positive meridian of K to a positive meridian, we can extend f to an orientation reversing homomorphism of (S^3, K) which reverses the orientation of K . Suppose

$$m : (S^3, K) \longrightarrow (S^3, K^*)$$

is a mirroring. Then $m \circ f$ is an orientation preserving homeomorphism which reverses the orientation of K . Thus K is equivalent to $-K^*$.

Next, we show “only if” part. By the assumption, there exists an orientation preserving homeomorphism

$$g : (S^3, K) \longrightarrow (S^3, K^*)$$

which reverses the orientation of K , and thus the composition $m^{-1} \circ g$ is an orientation reversing homeomorphism of (S^3, K) which reverses the orientation of K . $m^{-1} \circ g$ induces a map mapping a shadow coloring \mathcal{S}' of D with respect to $\mathcal{Q}(K)$ and Z to a shadow coloring \mathcal{S} of D with respect to $\mathcal{Q}(K)$ and Z . We assume $\Phi(\mathcal{S}', \text{vol}) = \text{vol}(S^3 \setminus K)$. Further $m^{-1} \circ g$ induces a chain map mapping a singular chain $\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r', x', y'}$ to a singular chain $\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}$, where $x', y' \in \mathcal{Q}(K)$ and $r' \in Z$ denote colors around a crossing point c with respect to \mathcal{S}' , and the correspondence of $x', y' \in \mathcal{Q}(K)$ and $r' \in Z$ with $x, y \in \mathcal{Q}(K)$ and $r \in Z$ is induced by $m^{-1} \circ g$. Since $m^{-1} \circ g$ reverses the orientation of S^3 ,

$$\left[\overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}}, \partial \overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}} \right] \in H_3(\overline{S^3 \setminus K}, \partial \overline{S^3 \setminus K})$$

is -1 times the fundamental class. \square

Now, let us consider another shadow coloring \mathcal{S} of a diagram D of K with respect to $\mathcal{Q}(K')$ and Z' , where K' is another oriented hyperbolic knot in S^3 , and Z' the set of homotopy classes of paths in $S^3 \setminus K'$. Then we also obtain a continuous map

$$f_{\mathcal{S}} : S^3 \setminus K \longrightarrow S^3 \setminus K'$$

by the same construction described in the previous section, although the range does not coincide with the domain. Further it is easy to see that

$$\begin{aligned} \Phi(\mathcal{S}, \text{vol}) &= \text{algvol}(f_{\mathcal{S}}(S^3 \setminus K)) \\ &= k \cdot \text{vol}(S^3 \setminus K') \quad (k \in \mathbb{Z}). \end{aligned}$$

In particular, the following theorem holds.

Theorem 4.2. *For each shadow coloring \mathcal{S} of a diagram D of K with respect to $\mathcal{Q}(-K)$ and Z , there exists $k \in \{-1, 0, 1\}$ such that*

$$\Phi(\mathcal{S}, \text{vol}) = k \cdot \text{vol}(S^3 \setminus K).$$

We omit the proof.

Theorem 4.3. *K is equivalent to $-K$ if and only if there exists a shadow coloring \mathcal{S} of a diagram D of K with respect to $\mathcal{Q}(-K)$ and Z satisfying*

$$\Phi(\mathcal{S}, \text{vol}) = \text{vol}(S^3 \setminus K).$$

Theorem 4.4. *K is equivalent to K^* if and only if there exists a shadow coloring \mathcal{S} of a diagram D of K with respect to $\mathcal{Q}(-K)$ and Z satisfying*

$$\Phi(\mathcal{S}, \text{vol}) = -\text{vol}(S^3 \setminus K).$$

Theorem 4.4 can be proved by the same line along the proof of Theorem 4.3. Thus we only prove Theorem 4.3.

Proof of Theorem 4.3. First, we show “if” part. By Thurston’s rigidity theorem again, there is an orientation preserving homeomorphism f of $S^3 \setminus K$ being homotopic to $f_{\mathcal{S}}$. Since $f_{\mathcal{S}}$ maps each positive meridian of K into a positive meridian of $-K$, we can also extend f to an orientation preserving homeomorphism of (S^3, K) which reverses the orientation of K . Thus K is equivalent to $-K$.

Next, we show “only if” part. By the assumption, there exists an orientation preserving homeomorphism

$$g : (S^3, K) \longrightarrow (S^3, K)$$

which reverses the orientation of K . g induces a map mapping a shadow coloring \mathcal{S}' of D with respect to $\mathcal{Q}(K)$ and Z to a shadow coloring \mathcal{S} of D with respect to $\mathcal{Q}(-K)$ and Z . We assume $\Phi(\mathcal{S}', \text{vol}) = \text{vol}(S^3 \setminus K)$. Further g induces a chain map mapping a singular chain $\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r', x', y'}$ to a singular chain $\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r, x, y}$, where $x', y' \in \mathcal{Q}(K)$ and $r' \in Z$ denote colors around a crossing point c with respect to \mathcal{S}' , and the

correspondence of $x', y' \in \mathcal{Q}(K)$ and $r' \in Z$ with $x, y \in \mathcal{Q}(-K)$ and $r \in Z$ is induced by g . Since g preserves the orientation of S^3 ,

$$\left[\overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r,x,y}}, \partial \overline{\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r,x,y}} \right] \in H_3(\overline{S^3 \setminus K}, \partial \overline{S^3 \setminus K})$$

is the fundamental class. \square

5. EXAMPLE

Let K be an oriented hyperbolic knot in S^3 , and \mathcal{S} a shadow coloring of a diagram D of K with respect to $\mathcal{Q}(K)$ and Z . We remark that even if we relocate the point p or q to another point p' or q' in S^3 along a path γ or δ respectively, the homology class of a singular chain $\sum_{c \in \mathcal{C}} \varepsilon(c) C_{r,x,y}$ with respect to \mathcal{S} does not change. Further if we choose p' or q' on K then there is a positive meridian w or z in $\mathcal{Q}(K)$ satisfying $\tilde{\gamma}(1) = w_\infty$ with $\tilde{\gamma}(0) = \tilde{p}$ or $\tilde{r} \cdot \delta(1) = z_\infty$ with $\tilde{r} \cdot \delta(0) = \tilde{p}$ respectively, where $r \cdot \delta$ denotes the composition of a representative path of r and δ , and each singular chain $C_{r,x,y}$ changes into a singular chain

$$\begin{aligned} C_{z,x,y}^w &= \{w_\infty, z_\infty, x_\infty, y_\infty\} + \{w_\infty, (z * x)_\infty, y_\infty, x_\infty\} \\ &\quad + \{w_\infty, ((z * x) * y)_\infty, (x * y)_\infty, y_\infty\} \\ &\quad + \{w_\infty, (z * y)_\infty, y_\infty, (x * y)_\infty\}. \end{aligned}$$

Thus the following theorem holds with a map

$$\text{vol}^w : \mathcal{Q}(K) \times \mathcal{Q}(K) \times \mathcal{Q}(K) \longrightarrow \mathbb{R}$$

defined by $\text{vol}^w(z, x, y) = \text{algvol}(C_{z,x,y}^w)$ with some $w \in \mathcal{Q}(K)$.

Theorem 5.1. *vol^w is an \mathbb{R} -valued quandle 2-cocycle with respect to $\mathcal{Q}(K)$. Further for each shadow coloring \mathcal{S} of a diagram D of K with respect to $\mathcal{Q}(K)$, there exists $k \in \{-1, 0, 1\}$ such that*

$$\Phi(\mathcal{S}, \text{vol}^w) = k \cdot \text{vol}(S^3 \setminus K).$$

We omit the proof. It is easy to see that similar theorems to Theorem 4.1, 4.2, 4.3 and 4.4 also hold with respect to this quandle 2-cocycle.

We close this paper by computing some elements of above quandle cocycle invariants for the figure eight knot. Associated with a diagram D of an oriented figure eight knot K , we choose Wirtinger generators

of $\pi_1(S^3 \setminus K)$ x , y , z , and w as depicted in Figure 8. Further we define a holonomy representation ρ or ρ_- of $\pi_1(S^3 \setminus K)$ or $\pi_1(S^3 \setminus -K)$ to satisfy the following equations respectively:

$$\begin{aligned} \rho(x) &= \begin{pmatrix} \frac{1-\sqrt{-3}}{2} & \frac{1+\sqrt{-3}}{2} \\ \frac{-1-\sqrt{-3}}{2} & \frac{3+\sqrt{-3}}{2} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 1 & \frac{1+\sqrt{-3}}{2} \\ 0 & 1 \end{pmatrix}, \\ \rho(z) &= \begin{pmatrix} \frac{3+\sqrt{-3}}{2} & \frac{1-\sqrt{-3}}{2} \\ 1 & \frac{1-\sqrt{-3}}{2} \end{pmatrix}, \text{ and } \rho(w) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \\ \rho_-(x^{-1}) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \rho_-(y^{-1}) = \begin{pmatrix} 1+\sqrt{-3} & \frac{1-\sqrt{-3}}{2} \\ \frac{3+3\sqrt{-3}}{2} & 1-\sqrt{-3} \end{pmatrix}, \\ \rho_-(z^{-1}) &= \begin{pmatrix} \frac{3+\sqrt{-3}}{2} & \frac{1-\sqrt{-3}}{2} \\ 1 & \frac{1-\sqrt{-3}}{2} \end{pmatrix}, \text{ and } \rho_-(w^{-1}) = \begin{pmatrix} \frac{1-\sqrt{-3}}{2} & \frac{1+\sqrt{-3}}{2} \\ \frac{-1-\sqrt{-3}}{2} & \frac{3+\sqrt{-3}}{2} \end{pmatrix}. \end{aligned}$$

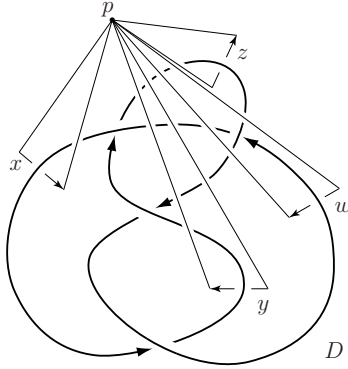


FIGURE 8. Wirtinger generators of $\pi_1(S^3 \setminus K)$

Example 5.2. For a shadow coloring \mathcal{S}_1 of D with respect to $\mathcal{Q}(K)$ depicted in the left-hand side of Figure 9,

$$\begin{aligned} \Phi(\mathcal{S}_1, \text{vol}^w) &= \text{algvol}(\{w_\infty, (y * z)_\infty, y_\infty, z_\infty\} - \{w_\infty, y_\infty, z_\infty, x_\infty\}) \\ &= \text{algvol}(\{0, \frac{-1+\sqrt{-3}}{2}, \infty, \frac{1+\sqrt{-3}}{2}\} - \{0, \infty, \frac{1+\sqrt{-3}}{2}, 1\}) \\ &= \text{vol}(S^3 \setminus K), \end{aligned}$$

where we use the upper half-space model of \mathbb{H}^3 .

Example 5.3. For a shadow coloring \mathcal{S}_2 of D with respect to $\mathcal{Q}(K)$ depicted in the right-hand side of Figure 9,

$$\begin{aligned}
\Phi(\mathcal{S}_2, \text{vol}^w) &= \text{algvol}(\{w_\infty, (y * w)_\infty, (y * z)_\infty, y_\infty\} \\
&\quad - \{w_\infty, (y * z)_\infty, y_\infty, z_\infty\}) \\
&= \text{algvol}(\{0, -1, \frac{-1+\sqrt{-3}}{2}, \infty\} - \{0, \frac{-1+\sqrt{-3}}{2}, \infty, \frac{1+\sqrt{-3}}{2}\}) \\
&= -\text{vol}(S^3 \setminus K).
\end{aligned}$$

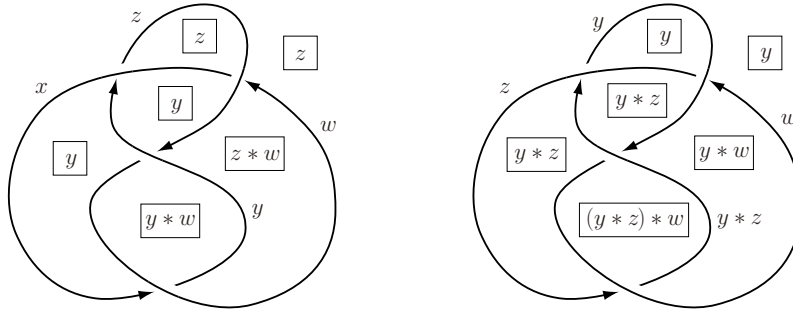


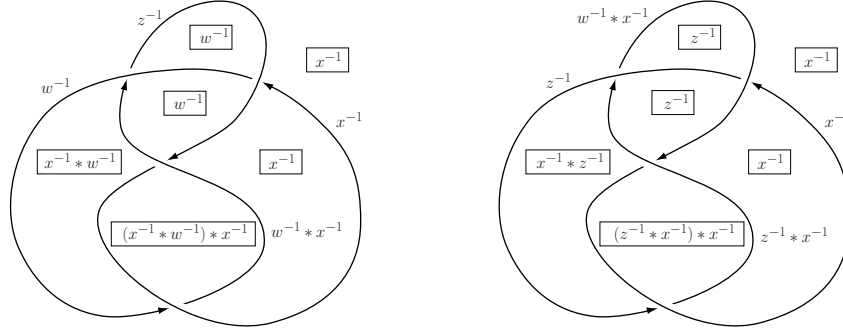
FIGURE 9. \mathcal{S}_1 (left) and \mathcal{S}_2 (right)

Example 5.4. For a shadow coloring \mathcal{S}_3 of D with respect to $\mathcal{Q}(-K)$ depicted in the left-hand side of Figure 10,

$$\begin{aligned}
\Phi(\mathcal{S}_3, \text{vol}^{x^{-1}}) &= \text{algvol}(\{x_\infty^{-1}, w_\infty^{-1}, z_\infty^{-1}, (w^{-1} * x^{-1})_\infty\} \\
&\quad - \{x_\infty^{-1}, (x^{-1} * w^{-1})_\infty, (w^{-1} * x^{-1})_\infty, w_\infty^{-1}\}) \\
&= \text{algvol}(\{0, 1, \frac{1+\sqrt{-3}}{2}, \infty\} - \{0, \frac{1-\sqrt{-3}}{2}, \infty, 1\}) \\
&= \text{vol}(S^3 \setminus K).
\end{aligned}$$

Example 5.5. For a shadow coloring \mathcal{S}_4 of D with respect to $\mathcal{Q}(-K)$ depicted in the right-hand side of Figure 10,

$$\begin{aligned}
\Phi(\mathcal{S}_4, \text{vol}^{x^{-1}}) &= \text{algvol}(\{x_\infty^{-1}, z_\infty^{-1}, (w^{-1} * x^{-1})_\infty, (z^{-1} * x^{-1})_\infty\} \\
&\quad - \{x_\infty^{-1}, (x^{-1} * z^{-1})_\infty, (z^{-1} * x^{-1})_\infty, z_\infty^{-1}\}) \\
&= \text{algvol}(\{0, \frac{1+\sqrt{-3}}{2}, \infty, \frac{-1+\sqrt{-3}}{2}\} \\
&\quad - \{0, \frac{\sqrt{-3}}{3}, \frac{-1+\sqrt{-3}}{2}, \frac{1+\sqrt{-3}}{2}\}) \\
&= -\text{vol}(S^3 \setminus K).
\end{aligned}$$

FIGURE 10. \mathcal{S}_3 (left) and \mathcal{S}_4 (right)

In conclusion, we have confirmed that the figure eight knot is invertible and positive/negative amphicheiral, as is well known.

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